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2001 J. Phys. A: Math. Gen. 34 845

(http://iopscience.iop.org/0305-4470/34/4/311)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **34** (2001) 845–859 www.iop.org/Journals/ja PII: S0305-4470(01)15484-3

Boundary states for a free boson defined on finite geometries

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Received 12 July 2000, in final form 17 October 2000

Abstract

Langlands recently constructed a map $\varphi \to |\mathfrak{x}(\varphi)\rangle$ that factorizes the partition function of a free boson on a cylinder with boundary condition given by two arbitrary functions φ_{B_1} and φ_{B_2} in the form $\langle\mathfrak{x}(\varphi_{B1})|q^{L_0+\bar{L}_0}|\mathfrak{x}(\varphi_{B2})\rangle$. We rewrite $|\mathfrak{x}(\varphi)\rangle$ in a compact form, getting rid of technical assumptions necessary in his construction. We show that this vector transforms properly under conformal transformations that preserve both the boundary and the reality conditions on the field φ. The vector $|\mathfrak{x}(\varphi)\rangle$ turns out to be the boundary state introduced by string theorists to compute corrections to closed-string amplitudes in non-trivial backgrounds. Dirichlet and Neumann states are written as a superposition of these boundary states.

PACS numbers: 1125H, 0530J, 1130

1. Introduction

Renormalization transformations are often defined on spaces of fields or parameters of statistical physics models. It is assumed that the existence of a non-trivial fixed point of these transformations requires that the space it acts on be infinite dimensional. Or at least that the physical relevance of such fixed points stems from the infinite number of degrees of freedom. Langlands [1] introduced a family of *finite* models inspired by percolation, each endowed with a renormalization transformation with a non-trivial fixed point. Numerical analysis shows that, already for the coarsest models in the family, the critical exponents bear some similarities with those accepted in the literature for percolation.

The calculation presented here is a step towards extending Langlands' construction for percolation to other models with interaction. In [2] Langlands calculated the partition function of the free boson on a cylinder with fixed boundary conditions. This partition function may be interpreted as the probability of measuring given restrictions of the boson at the cylinder extremities. Langlands argued that the space of such probability distributions might be a natural set upon which to construct a renormalization transformation. He showed how to construct vectors $|\mathfrak{x}_B\rangle$ in the Fock space that represent the restriction at a given extremity B in such a way that the partition function is simply the expectation value of the evolution operator between the two boundary states:

$$Z(\varphi|_{B_1}, \varphi|_{B_2}) = \langle \mathfrak{x}_{B_1}| q^{L_0 \oplus \bar{L}_0} |\mathfrak{x}_{B_2}\rangle. \tag{1}$$

(The notation will be clarified in the following.) This is nothing but the Feynman–Kac formula. The present paper goes back to this formula to make Langlands' expression for the state $|\mathfrak{x}_B\rangle$ more explicit. We should underline that it is a rather unusual use of the Feynman–Kac formula. While most of its applications involve the computation of a partition function from the evolution operator and boundary conditions, we use it as a tool to define boundary states in a chosen algebraic structure.

The answer for $|\mathfrak{x}_B\rangle$ is not new and can be found in various forms in the string theory literature (see, for instance, [10–12]). Let us review quickly how it appeared there. Early on came the necessity to compute open-string corrections to closed-string tree amplitudes in the presence of non-vanishing spacetime backgrounds of open-string fields. These corrections amount to gluing a cylinder to the tree-level process. The new boundary represents the creation out of the vacuum of some closed-string state, the cylinder is the propagation of that state from the boundary to the original process and its length is kept free to account for all moduli of the new worldsheet. A first step in this calculation is therefore to characterize the boundary state of this cylinder. In the absence of open-string backgrounds, this state must satisfy free boundary conditions: $\partial_+ x^\mu = -\partial_- x^\mu$, where x^μ, $\mu = 1, \ldots, D$ is the (first quantization) field associated with the spacetime coordinates of the string. In terms of creation and annihilation operators, these conditions require that the boundary state should be a (zero-eigenvalue) eigenstate of $(\mathfrak{a}_n^\mu - \bar{\mathfrak{a}}_{-n}^\mu)$. The bar distinguishes the right- from the left-moving sector. If open-string backgrounds are to be included, it is natural to construct a coherent state set in which each vector is a simultaneous eigenstate of $(\mathfrak{a}_n^\mu - \bar{\mathfrak{a}}_{-n}^\mu)$ with eigenvalues x_n^μ, for $\mu = 1, \ldots, D$ and $n \in \mathbb{N}$. It is by solving this infinite set of linear equations that string theorists arrived at the expression for the boundary states. Our path is quite different; out starting point is the partition function of a free boson with fixed boundary conditions that, in string theory, would be interpreted as the scattering amplitude between two classical states. Then we use the Feynman–Kac formula to find the boundary state.

Two comments could help distinguish our result from recent works in conformal field theory and quantum integrability. First, the boundary states computed here *are not* conformally invariant. Conformal boundary states were introduced and described for minimal models by Cardy [3]. Even though some particular conformal states (e.g. Dirichlet and Neumann states, see section 6) can be easily constructed out of them, the states $|\mathfrak{x}_B(\varphi)\rangle$ are not conformal in general. (They do have a natural transformation law under conformal maps. See section 5.) Second, the boundary states $|\mathfrak{x}_B\rangle$ are neither dynamical nor chosen to preserve an integrability property, like the S-matrix factorization, nor are such amplitudes between them expressible as a sum of Virasoro character with non-negative integer coefficients. As will be seen in section 3, the action integrated to obtain the partition function does not contain a boundary term. The results reported here are therefore not connected *a priori* to the body of work on quantum integrability of models on domains with boundary even though $|\mathfrak{x}(\varphi)\rangle$, for a specific φ, could lead to an integrable model. (The literature here is extensive; papers often referred to as seminal in this context are those of Ghoshal and Zamolodchikov [13] and of Callan and Klebanov [14].)

Our goal here is therefore to obtain the expression for the boundary states $|\mathfrak{x}_B\rangle$ in a novel way. We hope that the techniques described here can be extended to other models, e.g. the minimal ones. The paper is organized as follows. The notation is introduced in section 2. The computation of the partition function with boundary conditions is performed in the section 3. Sections 4 and 5 give a compact expression for $|\mathfrak{x}_B\rangle$ and show that it transforms properly under conformal transformations that preserve the boundary and the reality of the boson field. Section 6 expresses the (conformally invariant) Dirichlet and Neumann boundary states in terms of superpositions of $|\mathfrak{x}_B\rangle$.

2. Notation

The description of free bosons is based on the Heisenberg algebra and its representations. The generators are the creation ($\mathfrak{a}_{-k}$, $k > 0$), annihilation ($\mathfrak{a}_k$, $k > 0$) and central ($\mathfrak{a}_0$) operators, obeying the commutation rule

$$[\mathfrak{a}_n, \mathfrak{a}_m] = n\delta_{n+m,0}. \tag{2}$$

The Fock space $\mathcal{F}_\alpha$ is a highest-weight representation. The action of the generators on the highest-weight vector $|\alpha\rangle$ is given by

$$\mathfrak{a}_k|\alpha\rangle = 0 \qquad \forall k > 0 \tag{3}$$

$$\mathfrak{a}_0|\alpha\rangle = \alpha|\alpha\rangle \tag{4}$$

and physical states are generated by polynomials in $\mathfrak{a}_{-k}$, $k > 0$. A basis for $\mathcal{F}_\alpha$ is given by the vectors

$$|\alpha; n_1, n_2, \ldots\rangle = \mathfrak{a}_{-1}^{n_1}\mathfrak{a}_{-2}^{n_2}\cdots|\alpha\rangle \tag{5}$$

where the non-negative integers n_i are all zero except for finitely many. The inner product on $\mathcal{F}_\alpha$ is defined by

$$\begin{aligned}\langle\alpha'; n_1', n_2', \ldots|\alpha; n_1, n_2, \ldots\rangle &= \langle\alpha'|\cdots\mathfrak{a}_3^{n_3'}\mathfrak{a}_2^{n_2'}\mathfrak{a}_1^{n_1'}\mathfrak{a}_{-1}^{n_1}\mathfrak{a}_{-2}^{n_2}\mathfrak{a}_{-3}^{n_3}\cdots|\alpha\rangle \\ &= \delta_{\alpha,\alpha'}\left(\prod_{k=1}^{\infty} k^{n_k}\, n_k!\,\delta_{n_k,n_k'}\right).\end{aligned} \tag{6}$$

The elements (5) can thus be easily normalized to form an orthonormal basis.

The Hilbert space of the free boson is the direct sum of tensor products of the form $\mathcal{F}_\alpha\otimes\mathcal{F}_{\bar\alpha}$, $\mathcal{F}_\alpha$ and $\mathcal{F}_{\bar\alpha}$ characterizing modes in the holomorphic and antiholomorphic sectors, respectively. States in these tensor products are generated by the action of polynomials in $\mathfrak{a}_{-k}$ and $\bar{\mathfrak{a}}_{-k}$ on the highest-weight vector $|\alpha;\bar\alpha\rangle = |\alpha\rangle\otimes|\bar\alpha\rangle$. The generators $\mathfrak{a}_k$ are understood to act as $\mathfrak{a}_k\otimes 1$ and the $\bar{\mathfrak{a}}_k$ as $1\otimes\mathfrak{a}_k$.

Fock spaces are given by the structure of a Virasoro module by defining the conformal generators

$$\begin{aligned} L_n &= \tfrac{1}{2}\sum_{m\in\mathbb{Z}} :\mathfrak{a}_{n-m}\mathfrak{a}_m: \qquad n \neq 0 \\ L_0 &= \sum_{n>0}\mathfrak{a}_{-n}\mathfrak{a}_n + \tfrac{1}{2}\mathfrak{a}_0^2. \end{aligned} \tag{7}$$

The expression for L_0 implies that $\mathcal{F}_\alpha$ is a highest-weight module with highest weight $\alpha^2/2$. As will be described in the next section, the boson field is to be compactified on a circle of radius R. The pairs $(\alpha, \bar{\alpha})$ are then restricted to take the values

$$\alpha = \alpha_{u,v} = \left(\frac{u}{2R} + vR\right) \quad \rightarrow \quad h_{u,v} = \frac{1}{2}\left(\frac{u}{2R} + vR\right)^2 \tag{8}$$

$$\bar{\alpha} = \bar{\alpha}_{u,v} = \alpha_{u,-v} \quad \rightarrow \quad \bar{h}_{u,v} = \frac{1}{2}\left(\frac{u}{2R} - vR\right)^2 \tag{9}$$

with u and v integers and where $h_{u,v}$ and $\bar{h}_{u,v}$ are the values of $L_0 = L_0 \otimes 1$ and $\bar{L}_0 = 1 \otimes L_0$ acting on $\mathcal{F}_{\alpha_{u,v}} \otimes \mathcal{F}_{\bar{\alpha}_{u,v}}$. We will denote $\mathcal{F}_{\alpha_{u,v}}$ by $\mathcal{F}_{(u,v)}$ and $\mathcal{F}_{\bar{\alpha}_{u,v}}$ by $\bar{\mathcal{F}}_{(u,v)}$.

In his calculation, Langlands chose the Virasoro algebra Vir as the fundamental structure. He was able to construct explicitly the map $\mathfrak{x}$ for irreducible Verma modules over Vir. However, at $c = 1$, Verma modules over Vir are reducible whenever h_{uv} is equal to $m^2/4$ for some $m \in \mathbb{Z}$ [16]. (For example, this always happens for some integers u and v whenever $\sqrt{2}R$ is a rational number. Of course, when $u = v = 0$, it is reducible for any compactification radius R.) It was his suggestion that we look for an alternative definition that would encompass reducible cases. Using the Heisenberg algebra as the basic structure avoids this difficulty and also leads to an elegant form for $\mathfrak{x}$.

3. Explicit calculation of the partition function

We identify the cylinder with the quotient of the infinite strip $\ln q < \operatorname{Re} w < 0$, $0 < q < 1$, by the translations $w \rightarrow w + 2\pi i k$, $k \in \mathbb{Z}$. It can be mapped on the annulus $\mathcal{A}$ of centre 0, outer radius 1 and inner radius q by the conformal map $z = e^w$. The angle θ of the annular geometry parametrizes both extremities of the cylinder.

The partition function is defined as

$$\int \mathcal{D}\varphi e^{-\int_{\mathcal{A}} \mathcal{L}(\varphi)\, d^2 z} \tag{10}$$

where $\int_{\mathcal{A}}$ denotes the integration over the annulus, and the Lagrangian density is given by

$$\mathcal{L}(\varphi) = \partial_z \varphi \partial_{\bar{z}} \varphi. \tag{11}$$

The usual mode expansion of $\varphi(z, \bar{z})$ is

$$\varphi(z, \bar{z}) = \varphi_0 + a \ln z + b \ln \bar{z} + \sum_{n \neq 0} \left(\varphi_n z^n + \bar{\varphi}_n \bar{z}^n\right).$$

The restriction φ_{B_1} of this field to the inner circle where $z = q e^{i\theta}$ and $\bar{z} = q e^{-i\theta}$ is of the form

$$\varphi_{B1}(\theta) = \varphi_0 + (a + b) \ln q + i\theta(a - b) + \sum_{k \neq 0} b_k e^{ik\theta} \qquad b_{-k} = \bar{b}_k$$

and the restriction φ_{B_2} to the outer circle ($z = e^{i\theta}$, $\bar{z} = e^{-i\theta}$):

$$\varphi_{B2}(\theta) = \varphi_0 + i\theta(a - b) + \sum_{k \neq 0} a_k e^{ik\theta} \qquad a_{-k} = \bar{a}_k.$$

(The relationship between a_k, b_k and φ_n will be given below.) Since it is the field $e^{i\varphi/R}$ that really matters, φ need not be periodic but should only satisfy the milder requirement $\varphi(e^{2\pi i} z, e^{-2\pi i} \bar{z}) = \varphi(z, \bar{z}) + 2\pi v R$, $v \in \mathbb{Z}$. This statement is equivalent to the compactification of the field φ on a circle of radius R and implies that

$$a - b = -ivR \qquad v \in \mathbb{Z}.$$

The Lagrangian density does not depend on φ_0 and this constant may be set to zero. Therefore, only the difference of the constant terms in φ_{B_1} and φ_{B_2} remains. We choose to parametrize this difference by a real number $x \in [0, 2\pi R)$ and an integer $m \in \mathbb{Z}$:

$$-(a+b)\ln q = x + 2\pi m R.$$

The reason for this parametrization is again the compactification of φ: even though the various pairs $(\varphi_{B_1} + 2\pi m R, \varphi_{B_2})$, $m \in \mathbb{Z}$, will give different contributions to the functional integral, they all represent the same restriction of $e^{i\varphi/R}$ at the boundary.

We are interested in computing the partition function $Z(\varphi_{B_1}, \varphi_{B_2}) = Z(x, \{b_k\}, \{a_k\})$ defined as

$$Z(x, \{b_k\}, \{a_k\}) = \int_B \mathcal{D}\varphi e^{-\int_A \mathcal{L}(\varphi)\, d^2 z} \tag{12}$$

where $\int_B$ denotes the integration on the space of functions φ such that the restrictions of $e^{i\varphi/R}$ at the inner and outer boundaries coincide with $e^{i\varphi_{B_1}/R}$ and $e^{i\varphi_{B_2}/R}$. (The dependence on the compactification radius R is always implicit.) The decomposition of the field in a classical part verifying the boundary conditions and fluctuations vanishing at the extremities leads to

$$Z(x, \{b_k\}, \{a_k\}) = \Delta^{-1/2} Z_{\text{class}}(x, \{b_k\}, \{a_k\}). \tag{13}$$

The factor Δ is the ζ-regularization of the determinant for the annulus and is known to be (see, for example, [2, 4]):

$$\Delta^{-1/2} = (-i\tau)^{-1/2}\eta^{-1}(\tau) \qquad \text{with} \quad q = e^{i\pi\tau} \tag{14}$$

where $\eta(\tau) = e^{i\pi\tau/12}\prod_{m=1}^{\infty}(1 - e^{2im\pi\tau})$ is the Dedekind η function. The factor Z_{class} is the integration (sum) over all classical solutions compatible with the boundary conditions in the above sense. To obtain Z_{class} we solve the classical equations ($\partial_z\partial_{\bar z}\varphi = 0$) with boundary conditions given by $(\varphi_{B_1}, \varphi_{B_2})$. The condition at the outer circle ($z = e^{i\theta}, \bar z = e^{-i\theta}$) is $\varphi_n + \bar\varphi_{-n} = a_n$ and that at the inner one ($z = qe^{i\theta}, \bar z = qe^{-i\theta}$) is $q^n\varphi_n + q^{-n}\bar\varphi_{-n} = b_n$. The solution can be written as the sum

$$\varphi = a\ln z + b\ln\bar z + \tilde\varphi_1 + \tilde\varphi_2 \tag{15}$$

where the two function $\tilde\varphi_1$ and $\tilde\varphi_2$ are harmonic inside the annulus and take, respectively, the values φ_{B1} and 0 on the inner boundary and the values 0 and φ_{B2} on the outer one. These functions are

$$\tilde\varphi_1(z, \bar z) = \sum_{k\neq 0} \frac{b_k}{q^k - 1/q^k}(z^k - \bar z^{-k})$$

$$\tilde\varphi_2(z, \bar z) = \sum_{k\neq 0} \frac{a_k}{1/q^k - q^k}\left(\left(\frac{z}{q}\right)^k - \left(\frac{\bar z}{q}\right)^{-k}\right).$$

Hence the classical solution φ is completely determined by the data $(x, \{b_k\}, \{a_k\})$ up to the two integers $m, v \in \mathbb{Z}$ that determine a and b. The factor Z_{class} is consequently the sum

$$\sum_{m,v\in\mathbb{Z}} e^{-\int_A \mathcal{L}(\varphi)\, d^2 z}$$

where $\varphi_{(m,v)}$ is the solution (15) with $-(a+b)\ln q = x + 2\pi m R$ and $a - b = -ivR$.

Using this expression and the Poisson summation formula on the index m, Langlands [2] computed the desired partition function as the product

$$Z(x, \{b_k\}, \{a_k\}) = \Delta^{-1/2} Z_1(x) Z_2(\{b_k\}, \{a_k\}) \tag{16}$$

where

$$Z_1(x) = \sum_{u,v\in\mathbb{Z}} \mathrm{e}^{\mathrm{i}ux/R} q^{u^2/4R^2+v^2R^2} = \sum_{u,v\in\mathbb{Z}} \mathrm{e}^{\mathrm{i}x(\alpha_{u,v}+\bar{\alpha}_{u,v})}\, q^{h_{u,v}+\bar{h}_{u,v}} \tag{17}$$

and

$$Z_2(\{b_k\},\{a_k\}) = \prod_{k=1}^{\infty} \exp\left(-2k\left(\frac{1+q^{2k}}{1-q^{2k}}(a_k a_{-k} + b_k b_{-k}) - \frac{2q^k}{1-q^{2k}}(a_k b_{-k} + b_k a_{-k})\right)\right). \tag{18}$$

4. Explicit form of the boundary states

In this section we rewrite (16) as a sum over $u, v \in \mathbb{Z}$ of terms of the form

$$Z^{(u,v)}(x,\{b_k\},\{a_k\}) = \langle \mathfrak{r}^{(u,v)}(\varphi_{B1})|q^{L_0+\bar{L}_0}|\mathfrak{r}^{(u,v)}(\varphi_{B2})\rangle \tag{19}$$

with $|\mathfrak{r}^{(u,v)}(\varphi)\rangle \in \mathcal{F}_{(u,v)} \otimes \bar{\mathcal{F}}_{(u,v)}$ and where

$$Z^{(u,v)}(x,\{b_k\},\{a_k\}) = \Delta^{-1/2}\mathrm{e}^{\mathrm{i}ux/R} q^{u^2/4R^2+v^2R^2} Z_2(\{b_k\},\{a_k\}).$$

The goal is therefore to find a map $\mathfrak{r}^{(u,v)}$ such that (19) holds. To do so we will first set $Z_2(\{b_k\},\{a_k\})$ in the form

$$Z_2(\{b_k\},\{a_k\}) = \prod_{k=1}^{\infty} \sum_{m,n} B^k_{m,n} q^{k(m+n)} A^k_{m,n} \tag{20}$$

where $A^k_{m,n} = A^k_{m,n}(a_k, a_{-k})$ and $B^k_{m,n} = B^k_{m,n}(b_k, b_{-k})$ are functions of only two variables. With $Q = q^k$, $a_\pm = 2\mathrm{i}\sqrt{k}a_{\pm k}$ and $b_\pm = -2\mathrm{i}\sqrt{k}b_{\pm k}$ for $k \geqslant 1$, the terms $\Delta^{-1/2}Z_2$ of (16) are a product over k of

$$\mathrm{e}^{a_+a_-/2}\mathrm{e}^{b_+b_-/2}\frac{\mathrm{e}^{(a_+a_-Q^2+a_+b_-Q+b_+a_-Q+b_+b_-Q^2)/(1-Q^2)}}{1-Q^2} \tag{21}$$

up to a constant depending only on τ. The factors in front are clearly factorizable and can be absorbed in the definition of $A^k_{m,n}$ and $B^k_{n,m}$. The remaining mixed term can be developed as a power series in Q:

$$\sum_{i,j,k,l=0}^{\infty} \frac{(a_+a_- + b_+b_-)^i}{i!}\frac{(a_+b_-)^j(b_+a_-)^k}{j!k!}\frac{(i+j+k+l)!}{(i+j+k)!l!}Q^{2i+j+k+2l}. \tag{22}$$

To achieve the form (20), the coefficient of the term $q^{k(m+n)}$ (i.e. of $Q^{2i+j+k+2l}$ with $2i+j+k+2l = m+n$) in the above expression has to be the product of two functions, one of (a_k, a_{-k}), the other of (b_k, b_{-k}). We concentrate on the terms with $j \geqslant k$ and denote by $S_{m,n}$ the factor of $(a_+b_-)^{m-n}Q^{m+n}$ with $j-k = m-n$. The terms with $j < k$ are treated similarly. With the use of $x = a_+a_-$ and $y = b_+b_-$, $S_{m,n}$ can be written as

$$S_{m,n}(x,y) = \sum_{i+k+l=n} \frac{(x+y)^i(xy)^k}{i!(m-n+k)!k!}\frac{(m+k)!}{(m-n+i+2k)!l!}.$$

This function is clearly symmetric in x and y. Define

$$R_{m,n}(x) \equiv S_{m,n}(x,0)$$

and

$$T_{m,n} \equiv R_{m,n}(0) = S_{m,n}(0,0).$$

Casting (22) in the form (20) will only be possible if

$$S_{m,n}(x,y)T_{m,n} = R_{m,n}(x)R_{m,n}(y). \tag{23}$$

That this condition is verified is highly non-trivial. It was found in [2] that it is, although in disguised form, the Saalschütz identity [8]. We thus have the factorization if we define

$$A^k_{m,n}(a_k, a_{-k}) = \frac{R_{m,n}(x)}{\sqrt{T_{m,n}}} a_+^{m-n} \mathrm{e}^{a_+a_-/2} \qquad \text{if} \quad m \geqslant n \tag{24}$$

where

$$R_{m,n} = \sum_{i=0}^{n} \frac{m!x^i}{i!(m-n)!(m-n+i)!(n-i)!} \qquad m \geqslant n$$

and

$$T_{m,n} = \frac{m!}{n!((m-n)!)^2} \qquad m \geqslant n.$$

This expression for $R_{m,n}$ shows that it is related to the nth Laguerre polynomial of the $(m-n)$th kind by

$$R_{m,n}(x) = \frac{L_n^{(m-n)}(-x)}{(m-n)!} \qquad m \geqslant n. \tag{25}$$

A similar calculation leads to

$$R_{m,n}(x) = \frac{L_m^{(n-m)}(-x)}{(n-m)!} \qquad n \geqslant m. \tag{26}$$

Going back to the initial notation, we finally obtain the desired form with

$$A^k_{m,n}(a_k, a_{-k}) = \begin{cases} (2\mathrm{i}\sqrt{k}a_k)^{m-n}\sqrt{\dfrac{n!}{m!}}\mathrm{e}^{-2k|a_k|^2} L_n^{(m-n)}(4k|a_k|^2) & m \geqslant n \\ (2\mathrm{i}\sqrt{k}a_{-k})^{n-m}\sqrt{\dfrac{m!}{n!}}\mathrm{e}^{-2k|a_k|^2} L_m^{(n-m)}(4k|a_k|^2) & n \geqslant m \end{cases} \tag{27}$$

and $B^k_{m,n}(b_k, b_{-k}) = \overline{A^k_{m,n}(b_{-k}, b_k)}$. (An orientation on the boundary must be chosen to define the map $\mathfrak{x}$. For example, moving in the positive direction of the parameter θ should put the cylinder at one's left. This explains the interchange $b_k \leftrightarrow b_{-k}$ in the functions B.)

It is now straightforward to define the map from the boundary conditions to the Hilbert space. We have just shown that the contribution of the (u, v) sector to the partition function can be written as

$$Z^{(u,v)}(x, \{b_k\}, \{a_k\}) = q^{h_{u,v}+\bar{h}_{u,v}} \mathrm{e}^{\mathrm{i}x(\alpha_{u,v}+\bar{\alpha}_{u,v})} \prod_{k\in\mathbb{N}} \sum_{m,n=0}^{\infty} B^k_{m,n} q^{k(m+n)} A^k_{m,n}.$$

(Note that both sectors (u, v) and $(u, -v)$ contribute the same quantity to Z. There seems therefore to be a freedom to attach (u, v) to either $\mathcal{F}_{(u,v)} \otimes \bar{\mathcal{F}}_{(u,v)}$ or $\mathcal{F}_{(u,-v)} \otimes \bar{\mathcal{F}}_{(u,-v)}$. This choice is resolved in the next section.) Using the fact that

$$\langle u, v| \frac{(\mathfrak{a}_k^m \otimes \mathfrak{a}_k^n)\,(\mathfrak{a}_{-k'}^{m'} \otimes \mathfrak{a}_{-k'}^{n'})}{\sqrt{k^{m+n}k'^{m'+n'}m!n!m'!n'!}} |u, v\rangle = \delta_{k,k'}\delta_{m,m'}\delta_{n,n'}$$

we obtain

$$Z^{(u,v)}(x,\{b_k\},\{a_k\}) = q^{h_{u,v}+\bar{h}_{u,v}} \mathrm{e}^{\mathrm{i}x(\alpha_{u,v}+\bar{\alpha}_{u,v})} \prod_{k,k'} \sum_{m,m',n,n'} B^k_{m,n} q^{k(m+n)} A^{k'}_{m',n'} \delta_{k,k'}\delta_{m,m'}\delta_{n,n'} \tag{28}$$

$$= \langle \mathfrak{r}^{(u,v)}(x_1,\{b_k\})| q^{L_0 \oplus \bar{L}_0} |\mathfrak{r}^{(u,v)}(x_2,\{a_k\})\rangle \tag{29}$$

where

$$|\mathfrak{r}^{(u,v)}(x_2,\{a_k\})\rangle = \mathrm{e}^{\mathrm{i}x_2(\alpha_{u,v}+\bar{\alpha}_{u,v})} \prod_{k=1}^{\infty} \sum_{m,n=0}^{\infty} A^k_{m,n}(a_k,a_{-k}) \frac{\mathfrak{a}^m_{-k} \otimes \mathfrak{a}^n_{-k}}{\sqrt{k^{m+n} m!n!}} |u,v\rangle. \tag{30}$$

We have reintroduced, somewhat arbitrarily, the constant term x_2 in φ_{B2}. Again, only the difference $x = x_2 - x_1$ between the constant term x_2 in φ_{B2} and x_1 in φ_{B1} has a physical meaning. We now have an explicit form for the map $\mathfrak{r}$.

The vector $|\mathfrak{r}^{(u,v)}(x_2,\{a_k\})\rangle$ can be cast into a simpler form. With the help of the following recursion identities:

$$(n+1)L^{(m-(n+1))}_{n+1}(x) - [x\partial_x - x + (m-n)]L^{(m-n)}_n(x) = 0 \qquad m-1 \geqslant n \geqslant 0$$

and

$$L^{((m+1)-n)}_n(x) + [\partial_x - 1]L^{(m-n)}_n(x) = 0 \qquad m \geqslant n \geqslant 0$$

we can prove by induction on both indices that

$$A^k_{m,n} = \frac{(\partial_- + \frac{1}{2}a_+)^m (\partial_+ + \frac{1}{2}a_-)^n}{\sqrt{m!n!}} \mathrm{e}^{a_+ a_-/2} \tag{31}$$

where, we recall, $a_\pm = 2\mathrm{i}\sqrt{k} a_{\pm k}$ and $\partial_\pm = \frac{\partial}{\partial a_\pm}$. Defining $\alpha_k = \frac{1}{2}\mathrm{i}(-\partial_{-k} + 2ka_k)$, $\bar{\alpha}_k = \frac{1}{2}\mathrm{i}(-\partial_k + 2ka_{-k})$ and $\Omega_k = A^k_{0,0} = \mathrm{e}^{a_+ a_-/2} = \mathrm{e}^{-2k|a_k|^2}$, we obtain

$$A^k_{m,n} = \frac{\alpha^m_k \bar{\alpha}^n_k}{\sqrt{k^{m+n} m!n!}} \Omega_k. \tag{32}$$

The correspondence $\mathfrak{a}_{-k} \leftrightarrow \mathrm{i}\alpha_k$ and $\bar{\mathfrak{a}}_{-k} \leftrightarrow \mathrm{i}\bar{\alpha}_k$ induces an isomorphism with a subalgebra of the Heisenberg algebra since the α_ks and $\bar{\alpha}_k$s satisfy the commutation rules

$$[\alpha_n, \alpha_m] = -n\delta_{n+m,0} \qquad [\bar{\alpha}_n, \bar{\alpha}_m] = -n\delta_{n+m,0}$$
$$[\alpha_n, \bar{\alpha}_m] = 0.$$

If $|\alpha_{uv}\rangle \otimes |\bar{\alpha}_{u,v}\rangle$ is identified with $\Omega = \prod_k \Omega_k$ and α_0 (respectively $\bar{\alpha}_0$) is defined as acting by multiplication by $\alpha_{u,v}$ (respectively $\bar{\alpha}_{u,v}$), this correspondence can then be extended to an isomorphism of Heisenberg modules. Since the $\mathfrak{a}_{-k}$s and α_ks, $k > 0$, all commute with one another, we are able to write down an exponential form for the boundary state:

$$\begin{aligned}
|\mathfrak{r}^{(u,v)}(x_2,\{a_k\})\rangle &= \mathrm{e}^{\mathrm{i}x_2(\alpha_{u,v}+\bar{\alpha}_{u,v})} \prod_k \left\{ \sum_{m,n} \frac{(\alpha_k \mathfrak{a}_{-k})^m (\bar{\alpha}_k \bar{\mathfrak{a}}_{-k})^n}{m!n!k^m k^n} \Omega_k \right\} |u,v\rangle \\
&= \mathrm{e}^{\mathrm{i}x_2(\alpha_{u,v}+\bar{\alpha}_{u,v})} \prod_k \left\{ \mathrm{e}^{\alpha_k \mathfrak{a}_{-k}/k} \mathrm{e}^{\bar{\alpha}_k \bar{\mathfrak{a}}_{-k}/k} \right\} \Omega |u,v\rangle \\
&= \mathrm{e}^{\mathrm{i}x_2(\alpha_{u,v}+\bar{\alpha}_{u,v})} \prod_k \left\{ \mathrm{e}^{(\alpha_k \mathfrak{a}_{-k} + \bar{\alpha}_k \bar{\mathfrak{a}}_{-k})/k} \right\} \Omega |u,v\rangle \\
&= \mathrm{e}^{\mathrm{i}x_2(\alpha_{u,v}+\bar{\alpha}_{u,v})} \mathrm{e}^{\sum_{k\in\mathbb{N}} (\alpha_k \mathfrak{a}_{-k} + \bar{\alpha}_k \bar{\mathfrak{a}}_{-k})/k} \Omega |u,v\rangle.
\end{aligned} \tag{33}$$

Up to the factor $(-\mathrm{i}\tau)^{-1/2}\mathrm{e}^{-\mathrm{i}\pi\tau/12}$ the partition function takes the following form:

$$Z(x,\{b_k\},\{a_k\}) = \langle \mathfrak{x}(x_1,\{b_k\})|q^{L_0+\bar{L}_0}|\mathfrak{x}(x_2,\{a_k\})\rangle \tag{34}$$

in which we have defined

$$|\mathfrak{x}(x_2,\{a_k\})\rangle = \mathrm{e}^{\mathrm{i}x_2(\mathfrak{a}_0+\bar{\mathfrak{a}}_0)}\mathrm{e}^{\sum_{k=1}^{\infty}(\alpha_k\mathfrak{a}_{-k}+\bar{\alpha}_k\bar{\mathfrak{a}}_{-k})/k}\Omega|\Lambda\rangle \tag{35}$$

$$|\Lambda\rangle = \bigoplus_{u,v}|u,v\rangle \tag{36}$$

where the operators $\mathfrak{a}_0 = \mathfrak{a}_0\otimes 1$ and $\bar{\mathfrak{a}}_0 = 1\otimes\mathfrak{a}_0$ act as the identity times $\alpha_{u,v}$ and $\bar{\alpha}_{u,v}$ on $|u,v\rangle = |\alpha_{u,v}\rangle\otimes|\bar{\alpha}_{u,v}\rangle$. The boundary states $|\mathfrak{x}(x_2,\{a_k\})\rangle$ belong to the direct sum of Fock spaces $\bigoplus_{u,v}\mathcal{F}_{(u,v)}\otimes\bar{\mathcal{F}}_{(u,v)}$ or, more precisely, to the sum $\bigoplus_{u,v}(\mathcal{F}_{(u,v)}\otimes\bar{\mathcal{F}}_{(u,v)})^c$ of some completions that contains formal series like (35). In the next section, it will turn out to be useful to include the x dependence in Ω, which will then be denoted as $\Omega_{u,v}$, to highlight its sector:

$$\Omega_{u,v} = \mathrm{e}^{\mathrm{i}x(\alpha_{u,v}+\bar{\alpha}_{u,v})}\Omega. \tag{37}$$

Expression (35) can easily be put into a form that appeared in the string theory literature. Using the fact that, for $k\geqslant 1$,

$$\alpha_k\Omega = 2\mathrm{i}ka_k\Omega \qquad\text{and}\qquad \bar{\alpha}_k\Omega = 2\mathrm{i}ka_{-k}\Omega$$

and

$$\bar{\alpha}_k\exp(2\mathrm{i}a_k\mathfrak{a}_{-k})\,\Omega = (2\mathrm{i}ka_{-k}+\mathfrak{a}_{-k})\exp(2\mathrm{i}a_k\mathfrak{a}_{-k})\,\Omega$$

we can write

$$|\mathfrak{x}^{(u,v)}(\varphi)\rangle = \mathrm{e}^{\mathrm{i}x(\mathfrak{a}_0+\bar{\mathfrak{a}}_0)}\mathrm{e}^{\sum_{k\geqslant1}\bar{\alpha}_k\bar{\mathfrak{a}}_{-k}/k}\mathrm{e}^{\sum_{k\geqslant1}\alpha_k\mathfrak{a}_{-k}/k}\Omega|u,v\rangle \tag{38}$$

$$= \mathrm{e}^{\mathrm{i}x(\mathfrak{a}_0+\bar{\mathfrak{a}}_0)}\mathrm{e}^{\sum_{k\geqslant1}\bar{\alpha}_k\bar{\mathfrak{a}}_{-k}/k}\mathrm{e}^{\sum_{k\geqslant1}2\mathrm{i}a_k\mathfrak{a}_{-k}}\Omega|u,v\rangle \tag{39}$$

$$= \mathrm{e}^{\mathrm{i}x(\mathfrak{a}_0+\bar{\mathfrak{a}}_0)}\mathrm{e}^{\sum_{k\geqslant1}(2\mathrm{i}ka_{-k}+\mathfrak{a}_{-k})\bar{\mathfrak{a}}_{-k}/k}\mathrm{e}^{\sum_{k\geqslant1}2\mathrm{i}a_k\mathfrak{a}_{-k}}\Omega|u,v\rangle \tag{40}$$

$$= \mathrm{e}^{\mathrm{i}x(\mathfrak{a}_0+\bar{\mathfrak{a}}_0)}\mathrm{e}^{\sum_{k\geqslant1}(\frac{1}{k}\mathfrak{a}_{-k}\bar{\mathfrak{a}}_{-k}+2\mathrm{i}a_{-k}\bar{\mathfrak{a}}_{-k}+2\mathrm{i}a_k\mathfrak{a}_{-k}-2ka_ka_{-k})}|u,v\rangle \tag{41}$$

which is the form found in [11]. (It is equation (2.21) of that paper though the phase $\mathrm{e}^{\mathrm{i}x(\mathfrak{a}_0+\bar{\mathfrak{a}}_0)}$ is missing there. As mentioned in the introduction these authors obtained the vector by the solution of an eigensystem and a phase is therefore not important for them. It will be crucial to us in the next section. Otherwise the comparison goes as follows. Their Fourier coefficients x_n correspond to our a_n through the relation $x_n = -2\sqrt{n}a_n$, $n\geqslant 1$, and their generators a_n are $a_n = \mathrm{i}\mathfrak{a}_n/\sqrt{n}$, $a_{-n} = -\mathrm{i}\mathfrak{a}_{-n}/\sqrt{n}$, $n\geqslant 1$, and similarly for the antiholomorphic sector. The phase appears in [12].)

5. Conformal transformations and boundary states

Having found an explicit and concise form for the boundary states, we can now study their properties under conformal transformations. Of course, due to the analogy with the string theory result and the parametrization invariance there, the result will not come as a surprise. However, it is enlightening to state the exact requirements in the present context and to perform the computation with the simple expression (33).

Let g be an infinitesimal conformal transformation that leaves the boundary unchanged and G the corresponding element in the Virasoro algebra. The purpose of this section is to show that the action of g on the boundary condition φ and that of G on $|\mathfrak{x}(\varphi)\rangle$ commute:

$$|\mathfrak{x}(g\varphi)\rangle = G|\mathfrak{x}(\varphi)\rangle. \tag{42}$$

We first discuss the actions g and G and the correspondence between them.

One can easily convince oneself that the only infinitesimal conformal transformations that preserve the centre and radius of a circle in the complex plane are linear combinations of

$$(l_p - \bar{l}_{-p}) \qquad p \in \mathbb{Z} \tag{43}$$

where the conformal generators l_p and $\bar{l}_p$ are defined as $l_p = -z^{p+1}\partial_z$ and $\bar{l}_p = -\bar{z}^{p+1}\partial_{\bar{z}}$. Note that the subalgebra $\oplus_{p\in\mathbb{Z}}\mathbb{C}(L_p - \bar{L}_{-p}) \subset \mathrm{Vir} \otimes \overline{\mathrm{Vir}}$ is centreless and the mapping defined by $(l_p - \bar{l}_{-p}) \to (L_p - \bar{L}_{-p})$ of the boundary-preserving conformal transformation into $\mathrm{Vir} \otimes \overline{\mathrm{Vir}}$ is an isomorphism. However, the transformations $(l_p - \bar{l}_{-p})$, $p \neq 0$, do not preserve the reality condition imposed on the boundary functions. The generators $(l_p + \bar{l}_p)$ and $\mathrm{i}(l_p - \bar{l}_p)$ do. Both reality and geometry-preserving conditions are therefore satisfied by the infinitesimal transformations

$$g_0^{(1)} = 1 + \mathrm{i}\epsilon\left\{l_0 - \bar{l}_0\right\} \tag{44}$$

$$g_p^{(1)} = 1 + \mathrm{i}\epsilon\left\{(l_p + l_{-p}) - (\bar{l}_p + \bar{l}_{-p})\right\} \qquad p > 0 \tag{45}$$

and

$$g_p^{(2)} = 1 + \epsilon\left\{(l_p - l_{-p}) + (\bar{l}_p - \bar{l}_{-p})\right\} \qquad p > 0. \tag{46}$$

We shall show that (42) holds if $g_p^{(i)}$s are defined as above and the corresponding $G_p^{(i)}$s are taken to be

$$G_0^{(1)} = 1 + \mathrm{i}\epsilon\{L_0 - \bar{L}_0\} \tag{47}$$

$$G_p^{(1)} = 1 + \mathrm{i}\epsilon\left\{(L_p + L_{-p}) - (\bar{L}_p + \bar{L}_{-p})\right\} \tag{48}$$

and

$$G_p^{(2)} = 1 + \epsilon\left\{(L_p - L_{-p}) + (\bar{L}_p - \bar{L}_{-p})\right\}. \tag{49}$$

Since, for $p \neq 0$, we have

$$[g_p^{(1)} - 1, g_0^{(1)} - 1] = -\epsilon p(g_p^{(2)} - 1)$$

the property for the second family of transformations follows directly if it is proven to be true for the first one. The action of G on the right-hand side of (42) is simply left-multiplication. On the left-hand side, the action is defined as usual by $(g\varphi)(z, \bar{z}) = \varphi \circ g^{-1}(z, \bar{z})$. We first study the case $p > 0$. For $p = 0$, the particularity of the Sugawara construction will modify the analysis. We will end this section by examining this case.

Let us first compute $|\mathfrak{x}(g_p\varphi)\rangle$ with $g_p = g_p^{(1)}$. Note that, due to the use of the Poisson summation formula to obtain (17), the constant m in $-(a+b)\ln q = x + 2\pi m R$ is no longer well defined in the sector (u, v). However, the difference $(a-b)$ still is. Only $a-b$ will appear in the variation $g_p\varphi$. As observed in the previous paragraph, the two contributions $Z^{(u,v)}$ and $Z^{(u,-v)}$ are equal. It turns out that equation (42) holds when the functions φ with a given v are mapped into the sectors $\mathcal{F}_{(u,v)} \otimes \bar{\mathcal{F}}_{(u,v)}$, $u \in \mathbb{Z}$. (For the other choice $\mathcal{F}_{(u,-v)} \otimes \bar{\mathcal{F}}_{(u,-v)}$, the actions g and G fail to commute.) The function on the boundary must have the form

$$\varphi(\theta) = x + vR\theta + \sum_{k>0}(a_k \mathrm{e}^{ik\theta} + a_{-k}\mathrm{e}^{-\mathrm{i}k\theta})$$

or, equivalently

$$\varphi(z,\bar z) = x + (a\ln z + b\ln\bar z) + \sum_{k>0}(a_k z^k + \bar a_k \bar z^k)$$

with $z = \mathrm{e}^{\mathrm{i}\theta}$ and $\bar z = \mathrm{e}^{-\mathrm{i}\theta}$ and the reality condition $a_{-k} = \bar a_k$. A direct calculation gives

$$g_p\varphi = \tilde x + vR\theta + \sum_{k>0}(c_k \mathrm{e}^{\mathrm{i}k\theta} + \bar c_k \mathrm{e}^{-\mathrm{i}k\theta})$$

where

$$c_k = a_k + \mathrm{i}\epsilon\left((k+p)a_{k+p} + (k-p)a_{k-p}\right) + \epsilon vR\delta_{k,p} \tag{50}$$

$$\bar c_k = \bar a_k - \mathrm{i}\epsilon\left((k+p)\bar a_{k+p} + (k-p)\bar a_{k-p}\right) + \epsilon vR\delta_{k,p} \tag{51}$$

$$\tilde x = x + \mathrm{i}\epsilon p(a_p - \bar a_p). \tag{52}$$

One can see that the reality condition imposed on φ is indeed preserved. Since

$$|\mathfrak{x}^{(u,v)}(\tilde x,\{c_k\})\rangle = \left(\mathrm{e}^{\sum_{k>0}(\alpha_k\mathfrak{a}_{-k}+\bar\alpha_k\bar{\mathfrak{a}}_{-k})/k}\Omega_{u,v}\right)\Big|_{g_p\varphi}|u,v\rangle \tag{53}$$

$$= \mathrm{e}^{\sum_{k>0}(\tilde\alpha_k\mathfrak{a}_{-k}+\tilde{\bar\alpha}_k\bar{\mathfrak{a}}_{-k})/k}\tilde\Omega_{u,v}|u,v\rangle \tag{54}$$

where we have defined

$$\tilde\alpha_k = \tfrac{1}{2}\mathrm{i}\left(-\frac{\partial}{\partial\bar c_k} + 2kc_k\right) \tag{55}$$

$$\tilde{\bar\alpha}_k = \tfrac{1}{2}\mathrm{i}\left(-\frac{\partial}{\partial c_k} + 2k\bar c_k\right) \tag{56}$$

$$\tilde\Omega_{u,v} = \mathrm{e}^{\mathrm{i}\tilde x(\alpha_{u,v}+\bar\alpha_{u,v})}\mathrm{e}^{-2\sum_{k>0}kc_k\bar c_k} \tag{57}$$

a first step is to express $\tilde\alpha_k$, $\tilde{\bar\alpha}_k$ and $\tilde\Omega_{u,v}$ in terms of x and the a_ks. This can be easily achieved. The expressions for c_k and $\bar c_k$ given above can be inverted in order to obtain closed-form expressions for a_k and $\bar a_k$. It is then a simple exercise to show that

$$\tilde\alpha_k = \tfrac{1}{2}\mathrm{i}\left(-\frac{\partial}{\partial\bar c_k} + 2kc_k\right) = \begin{cases}\alpha_k + \mathrm{i}\epsilon k(\alpha_{k+p}+\alpha_{k-p}) & k\neq p\\ \alpha_p + \mathrm{i}\epsilon p(\alpha_{2p}+\alpha_{uv}) & k = p\end{cases}$$

$$\tilde{\bar\alpha}_k = \tfrac{1}{2}\mathrm{i}\left(-\frac{\partial}{\partial c_k} + 2k\bar c_k\right) = \begin{cases}\bar\alpha_k - \mathrm{i}\epsilon k(\bar\alpha_{k+p}+\bar\alpha_{k-p}) & k\neq p\\ \bar\alpha_k - \mathrm{i}\epsilon p(\bar\alpha_{2p}+\bar\alpha_{uv}) & k = p.\end{cases}$$

Using these expressions, the functional $\tilde\Omega_{u,v} = \Omega_{u,v}\big|_{g_p\varphi}$ can be expressed in terms of the original variables. A careful treatment of the infinite sums leads to

$$\tilde\Omega_{u,v} = \left(1 + \mathrm{i}\epsilon\left(\frac{1}{2}\sum_{0<k<p}(\alpha_{p-k}\alpha_k - \bar\alpha_{p-k}\bar\alpha_k) + \alpha_p\alpha_{u,v} - \bar\alpha_p\bar\alpha_{u,v}\right)\right)\Omega_{u,v}. \tag{58}$$

One subtlety has to be noted in expanding $\exp\sum_k(\tilde\alpha_k\mathfrak{a}_{-k} + \tilde{\bar\alpha}_k\bar{\mathfrak{a}}_{-k})$ to first order in ϵ. Even though the $\mathfrak{a}_{-k}$s and $\bar{\mathfrak{a}}_{-k}$s ($k \geqslant 1$) all commute, the transformed $\tilde\alpha_k$ and $\tilde{\bar\alpha}_k$ may contain αs with negative indices. If one writes

$$A = \sum_{k\geqslant 1}\frac{1}{k}(\alpha_k\mathfrak{a}_{-k} + \bar\alpha_k\bar{\mathfrak{a}}_{-k})$$

$$B = \sum_{k\geqslant 1}((\alpha_{k+p}+\alpha_{k-p})\mathfrak{a}_{-k} - (\bar\alpha_{k+p}+\bar\alpha_{k-p})\bar{\mathfrak{a}}_{-k})$$

where α_0 ($\bar{\alpha}_0$) is understood as the multiplication by α_{uv} ($\bar{\alpha}_{uv}$), then their commutator is

$$[A, B] = -\sum_{0<k<p} \left(\mathfrak{a}_{-(p-k)}\mathfrak{a}_{-k} - \bar{\mathfrak{a}}_{-(p-k)}\bar{\mathfrak{a}}_{-k}\right).$$

It commutes with the $\mathfrak{a}_{-k}$s and $\bar{\mathfrak{a}}_{-k}$s, $k > 0$, and the new exponential is simply $\mathrm{e}^{A+\mathrm{i}\epsilon B} = \mathrm{e}^{A}(1 + \mathrm{i}\epsilon(B - \frac{1}{2}[A, B]))$. Finally, we can write $|\mathfrak{x}^{(u,v)}(g_p\varphi)\rangle$ to first order in ϵ as

$$\begin{aligned}
|\mathfrak{x}^{(u,v)}(g_p\varphi)\rangle &= \mathrm{e}^{\sum_k(\tilde{\alpha}_k\mathfrak{a}_{-k}+\tilde{\bar{\alpha}}_k\bar{\mathfrak{a}}_{-k})/k}\tilde{\Omega}_{u,v}|u, v\rangle \\
&= \mathrm{e}^{\sum_k(\alpha_k\mathfrak{a}_{-k}+\bar{\alpha}_k\bar{\mathfrak{a}}_{-k})/k}\Bigg(1 + \mathrm{i}\epsilon\Bigg(\sum_{k>0}\Bigg((\alpha_{k+p} + \alpha_{k-p})\mathfrak{a}_{-k} - (\bar{\alpha}_{k+p} + \bar{\alpha}_{k-p})\bar{\mathfrak{a}}_{-k}\Bigg) \\
&\quad +\alpha_p\mathfrak{a}_0 - \bar{\alpha}_p\bar{\mathfrak{a}}_0 \\
&\quad +\tfrac{1}{2}\sum_{0<k<p}(\alpha_{p-k}\alpha_k - \bar{\alpha}_{p-k}\bar{\alpha}_k + \mathfrak{a}_{-(p-k)}\mathfrak{a}_{-k} - \bar{\mathfrak{a}}_{-(p-k)}\bar{\mathfrak{a}}_{-k})\Bigg)\Bigg)\Omega_{u,v}|u, v\rangle \\
&= \mathrm{e}^{\sum_k(\alpha_k\mathfrak{a}_{-k}+\bar{\alpha}_k\bar{\mathfrak{a}}_{-k})/k}\Bigg(1 + \mathrm{i}\epsilon\sum_{k\geqslant 0}(\alpha_{k+p}\mathfrak{a}_{-k} + \alpha_k\mathfrak{a}_{-k-p} - \bar{\alpha}_{k+p}\bar{\mathfrak{a}}_{-k} - \bar{\alpha}_k\bar{\mathfrak{a}}_{-k-p}) \\
&\quad +\frac{\mathrm{i}\epsilon}{2}\sum_{0<k<p}(\alpha_{p-k}\alpha_k - \bar{\alpha}_{p-k}\bar{\alpha}_k + \mathfrak{a}_{-(p-k)}\mathfrak{a}_{-k} - \bar{\mathfrak{a}}_{-(p-k)}\bar{\mathfrak{a}}_{-k})\Bigg)\Omega_{u,v}|u, v\rangle
\end{aligned} \tag{59}$$

where, for the last equality, we took advantage of $\alpha_{-k}\Omega_{uv} = \bar{\alpha}_{-k}\Omega_{uv} = 0$ if $k > 0$.

The computation of the right-hand side of (42) is more direct. First write L_p and L_{-p}, $p > 0$, as

$$L_p = \sum_{k\geqslant 0}\mathfrak{a}_{-k}\mathfrak{a}_{k+p} + \tfrac{1}{2}\sum_{0<k<p}\mathfrak{a}_{p-k}\mathfrak{a}_k \tag{60}$$

$$L_{-p} = \sum_{k\geqslant 0}\mathfrak{a}_{-k-p}\mathfrak{a}_k + \tfrac{1}{2}\sum_{0<k<p}\mathfrak{a}_{-(p-k)}\mathfrak{a}_{-k} \tag{61}$$

and similarly for $\bar{L}_p$ and $\bar{L}_{-p}$. Second, note that for $k \geqslant 0$

$$\mathfrak{a}_k|\mathfrak{x}^{(u,v)}(\varphi)\rangle = \alpha_k|\mathfrak{x}^{(u,v)}(\varphi)\rangle \qquad \text{and} \qquad \bar{\mathfrak{a}}_k|\mathfrak{x}^{(u,v)}(\varphi)\rangle = \bar{\alpha}_k|\mathfrak{x}^{(u,v)}(\varphi)\rangle. \tag{62}$$

Then

$$\begin{aligned}
G_p|\mathfrak{x}^{(u,v)}(\varphi)\rangle &= \left(1 + \mathrm{i}\epsilon((L_p + L_{-p}) - (\bar{L}_p + \bar{L}_{-p}))\right)|\mathfrak{x}^{(u,v)}(\varphi)\rangle \\
&= \mathrm{e}^{\sum_k(\alpha_k\mathfrak{a}_{-k}+\bar{\alpha}_k\bar{\mathfrak{a}}_{-k})/k}\Bigg(1 + \mathrm{i}\epsilon\sum_{k\geqslant 0}(\alpha_{k+p}\mathfrak{a}_{-k} + \alpha_k\mathfrak{a}_{-k-p} - \bar{\alpha}_{k+p}\bar{\mathfrak{a}}_{-k} - \bar{\alpha}_k\bar{\mathfrak{a}}_{-k-p}) \\
&\quad +\frac{\mathrm{i}\epsilon}{2}\sum_{0<k<p}(\alpha_{p-k}\alpha_k - \bar{\alpha}_{p-k}\bar{\alpha}_k + \mathfrak{a}_{-(p-k)}\mathfrak{a}_{-k} - \bar{\mathfrak{a}}_{-(p-k)}\bar{\mathfrak{a}}_{-k})\Bigg)\Omega_{u,v}|u, v\rangle \\
&= |\mathfrak{x}^{(u,v)}(g_p\varphi)\rangle.
\end{aligned} \tag{63}$$

We have thus established the desired property for $p > 0$.

The transformation $g_0^{(1)} = 1 + \mathrm{i}\epsilon(l_0 - \bar{l}_0)$ is nothing but an infinitesimal rotation. The Gaussian terms are invariant under these transformations, because Fourier coefficients only pick up a phase. Hence $\Omega_{u,v}\big|_{g_0^{(1)}\varphi} = (1 + \mathrm{i}\epsilon v R(\alpha_{u,v} + \bar{\alpha}_{u,v}))\ \Omega_{u,v}\big|_{\varphi}$. The computation of $|\mathfrak{x}^{(u,v)}(g_0^{(1)}\varphi)\rangle$ is straightforward and one obtains

$$\begin{aligned}
|\mathfrak{x}^{(u,v)}(g_0^{(1)}\varphi)\rangle &= \mathrm{e}^{\sum_k(\alpha_k\mathfrak{a}_{-k}+\bar{\alpha}_k\bar{\mathfrak{a}}_{-k})/k} \\
&\quad\times\left(1 + \mathrm{i}\epsilon\sum_{k>0}\left(\alpha_k\mathfrak{a}_{-k} - \bar{\alpha}_k\bar{\mathfrak{a}}_{-k}\right) + \mathrm{i}\epsilon v R(\alpha_{u,v} + \bar{\alpha}_{u,v})\right)\Omega_{u,v}|u, v\rangle \\
&= \mathrm{e}^{\sum_k(\alpha_k\mathfrak{a}_{-k}+\bar{\alpha}_k\bar{\mathfrak{a}}_{-k})/k} \\
&\quad\times\left(1 + \mathrm{i}\epsilon\sum_{k>0}\left(\alpha_k\mathfrak{a}_{-k} - \bar{\alpha}_k\bar{\mathfrak{a}}_{-k}\right) + \mathrm{i}\epsilon(h_{u,v} - \bar{h}_{u,v})\right)\Omega_{u,v}|u, v\rangle.
\end{aligned} \tag{64}$$

The action of G_0 is also easy. Using the definition of L_0 and $\bar{L}_0$ in (7) and the eigenvalues of $\mathfrak{a}_0$ and $\bar{\mathfrak{a}}_0$ in (8), we obtain the desired equality: $|\mathfrak{x}^{(u,v)}(g_0\varphi)\rangle = G_0|\mathfrak{x}^{(u,v)}(\varphi)\rangle$. This completes the proof.

6. Dirichlet and Neumann boundary states as superpositions of $|\mathfrak{x}(\varphi)\rangle$

Dirichlet and Neumann boundary states for the free boson are among the simplest conformally invariant boundary states. Up to an overall factor their expressions are [11, 15]

$$\begin{aligned}
|D\rangle &= \sum_{u\in\mathbb{Z}} \mathrm{e}^{\mathrm{i}xu/R}\prod_{k\in\mathbb{N}}\exp\left(\frac{1}{k}\mathfrak{a}_{-k}\bar{\mathfrak{a}}_{-k}\right)|u, 0\rangle \\
|N\rangle &= \sum_{v\in\mathbb{Z}} \mathrm{e}^{2\mathrm{i}xvR}\prod_{k\in\mathbb{N}}\exp\left(-\frac{1}{k}\mathfrak{a}_{-k}\bar{\mathfrak{a}}_{-k}\right)|0, v\rangle.
\end{aligned} \tag{65}$$

The sums are not relevant for the discussion of conformal invariance. They will be left aside in the remaining of the section. (Note that these states are actually one-parameter families labelled by x.) It is an interesting exercise to express them as superpositions of the boundary states $|\mathfrak{x}^{(u,v)}(\{a_k\})\rangle$. We can solve for both simultaneously by introducing $|B_t\rangle = \exp(t\sum_k \frac{1}{k}\mathfrak{a}_{-k}\bar{\mathfrak{a}}_{-k})|u, v\rangle$ and looking for a function $\psi_t(\{a_k\})$ such that

$$|B_t\rangle = \int \psi_t(\{a_k\})|\mathfrak{x}^{(u,v)}(\{a_k\})\rangle\prod_{k\geqslant 1}\mathrm{d}a_k\,\mathrm{d}a_{-k}. \tag{66}$$

The particular values $t = +1$ and -1 correspond to the desired states $|D\rangle$ and $|N\rangle$.

Due to the specific form of (30), it is natural to look for ψ_t of the form $\psi_t(\{a_k\}) = \prod_{k\geqslant 1}\psi_t^k(a_k, a_{-k})$. Then (66) simply requires that

$$\int \psi_t^k(a_k, a_{-k})A_{m,n}^k(a_k, a_{-k})\,\mathrm{d}a_k\,\mathrm{d}a_{-k} = \delta_{mn}t^m \qquad \text{for}\quad k\geqslant 1 \text{ and } m, n\geqslant 0. \tag{67}$$

By choosing ψ_t^k to depend only on $|a_k|$, the integral automatically generate a δ_{mn}. The problem reduces to finding a function $\psi_t^k(a_k, a_{-k}) = \phi^k(t, x_k = 4k|a_k|^2)$ that satisfies

$$\frac{\pi}{4k}\int_0^\infty \mathrm{e}^{-x_k/2}\phi^k(t, x_k)L_m^{(0)}(x_k)\,\mathrm{d}x_k = t^m \tag{68}$$

where we have used the explicit form (27) for the $A_{m,n}^k$. Because the generating function for Laguerre polynomials is

$$f(x, t) = \sum_{m=0}^{\infty} L_m^{(0)}(x)t^m = \frac{\mathrm{e}^{-xt/(1-t)}}{1-t}$$

and their orthogonality relation $\int_0^\infty \mathrm{e}^{-x} L_m^{(0)}(x) L_n^{(0)}(x)\,\mathrm{d}x = \delta_{mn}$, the final answer is easily seen to be

$$\phi^k(t, x_k) = \frac{4k}{\pi} \mathrm{e}^{-x_k/2} f(x_k, t).$$

The limit of $\phi^k(t, x)$ as $t \to +1^-$ is not well defined but that of the integrals

$$\lim_{t \to +1^-} \int_0^\infty f(x, t) A(x)\,\mathrm{d}x$$

is for any function $A(x)$ in the space of functions spanned by $x^n \mathrm{e}^{bx}$, for all $b \in \mathbb{R}$ and $n \geqslant 0$. The limit of the integrals is then $A(0)$ and corresponds to the Dirichlet conditions $a_k = 0$ for all $k \neq 0$. The limit of $\phi^k(t, x)$ as $t \to -1$ is a constant and corresponds to the free boundary (Neumann) condition. Note that one also obtains the expansion of the vacuum ($\phi^k(0, x_k) = \frac{4k}{\pi} \mathrm{e}^{-x_k/2}$).

Only the values $t = \pm 1$ of $|B_t\rangle$ are conformally invariant. The requirement $(L_p - \bar{L}_{-p})|B_t\rangle = 0$ is, for $p > 0$:

$$\left(\prod \exp\left(\frac{1}{k} t \mathfrak{a}_{-k} \bar{\mathfrak{a}}_{-k}\right)\right) \left((\alpha_{uv} t - \bar{\alpha}_{u,v}) \bar{\mathfrak{a}}_{-p} + \tfrac{1}{2}(t^2 - 1) \sum_{0<k<p} \bar{\mathfrak{a}}_{-(p-k)} \bar{\mathfrak{a}}_{-k}\right) |uv\rangle = 0$$

and is satisfied only for $t = \pm 1$. Moreover, if $t = +1$ (-1), the integer v (u) must vanish in accordance with the sums in (65). The family $|B_t\rangle$ therefore does not include the family of conformally invariant boundary states introduced by Callan and Klebanov in [14].

7. Concluding remarks

This simple yet quite instructive calculation gives an example of a conformal theory with nonconformally invariant boundary conditions. Can the map $\varphi \to |\mathfrak{x}(\varphi)\rangle$ for the free boson be used to investigate minimal models with general boundary conditions? It is well known that minimal models can be constructed from the $c = 1$ conformal field theory, using the Coulomb gas technique. This was done successfully on the plane by Dotsenko and Fateev [5, 6] and on the torus by Felder [7]. This might be one path to constructing the map for these models.

Langlands and the two authors have recently studied numerically the statistical distribution of the Fourier coefficients of a field defined for the Ising model [9]. This distribution is more intricate than the boson's as the Fourier coefficients of the field at one boundary do not appear now to be mutually independent. The map $\varphi \to |\mathfrak{x}(\varphi)\rangle$, if it exists for the Ising model, might be a rich object.

Acknowledgments

The authors would like to thank Robert Langlands for explaining his original construction and stressing its limitations, Frédéric Lesage and Philippe Zaugg for helpful discussions and Rafael Nepomechie for bringing to our attention the papers in string theory relevant to this paper. We also thank both referees for their helpful comments. MAL gratefully acknowledges fellowships from the NSERC Canada Scholarships Programme and the Celanese Foundation, and YSA support from NSERC (Canada) and FCAR (Québec).

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